God and Mathematical Objects*

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1 Introduction: Models and Modernism

Postmodernists typically question the very coherence of claims to knowledge of "objective" reality, knowledge that is not at least implicitly qualified by reference to culture and context. There are a number of important insights behind the postmodernist's skepticism, notably the tight connection between claims of objective knowledge and the exercise of power.[1] However, certain domains of knowledge seem less suited to the postmodernist critique than others. The chief example is mathematics. One can imagine a number of reasons for this. First, in contrast to, say, physics, with its sweeping visions of the nature and origins of the universe, or genetics and its promise of power over disease, intelligence, and the like, it is far from clear how mathematics (of itself, anyway) could provide any great leverage for the exercise of power. Second, because of its nonempirical nature, rather than to subject it to critique, it is easier, perhaps, simply to discount claims of objectivity. Mathematics, the sophisticated thinker might argue, is not obviously *about* anything at all, and is therefore simply its own self-contained discourse with its own set of rules, in which a select group of practitioners engage — a view of discourse in general toward which the postmodernist is already antecedently inclined.

Such views of mathematics, however, do not comport well with the striking applicability of mathematics to our scientific understanding of the world. Nor does it do justice to the common intuition that mathematical truth is stable and objective, independent of any particular human mind — on the face of it, it is just objectively and eternally true that 7+5=12. Of course, many postmodernists would argue that such considerations beg all of the big questions at the outset. For belief in the privileged status of scientific knowledge, and intuitions about objective and eternal truth, simply indicate an unreflective acceptance of the central themes of modernity, and uncritical acceptance of the modern "grand narrative" that casts humans essentially as autonomous rational knowers and science as their primary epistemological instrument. Perhaps so. But what is supposed to follow? The postmodernist does not argue that the great modernist themes are *false*. Indeed, she cannot so argue, at least not directly. For that would play directly into the modernist's hands, as it would itself appear

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[1] See, e.g., M. Foucault, *Power/Knowledge: Selected Interviews and Other Writings* (New York: Pantheon, 1980).

to be a claim to objective truth, one in need of some sort of rational (i.e., broadly speaking, scientific) justification. The most that the postmodernist can offer (more along the lines of therapy than argument) is to point out the inability of any grand narrative — science in particular — to justify its own first principles,[2] and to point out the ease with which grand narratives slip into inflexible dogma that is wielded by those with power to further their own ends.[3] But the inability to justify first principles is hardly a revelation. That, after all, is what makes them first principles. The most that follows from this observation, and the tendency of grand narratives to serve dubious interests, is that one's acceptance of the grand themes of modernity should be appropriately tempered with a certain philosophical humility: philosophy is hard, and few philosophical theses deserve anything more than tentative, qualified acceptance. Beyond this, postmodernism gives us no cogent reason to abandon the modernist presuppositions of objective truth and its knowability — in particular, with regard to mathematics.

To this end, then, in this chapter I develop an unabashedly modernist model of the subject matter of mathematics and argue that it provides a reasonable ground for the possibility of mathematical knowledge. Indeed, I pull out all the modernist stops and place God — perhaps

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[2] This seems to be what Jean-Francois Lyotard is getting at in the following passages from *The Postmodern Condition* (Minneapolis: University of Minnesota Press, 1984), p. 29:

> Scientific knowledge cannot know and make known that it is the true knowledge without resorting to the other, narrative, kind of knowledge, which from its point of view is no knowledge at all. Without such recourse it would be in the position of presupposing its own validity and would be stooping to what it condemns: begging the question, proceeding on prejudice. But does it not fall into the same trap by using narrative as its authority?

And again:

> With modern science … [it] is recognized that the conditions of truth, in other words, the rules of the game of science, and immanent in that game, that they can only be established within the bonds of a debate that is already scientific in nature, and that there is no other proof that the rules are good than the consensus extended to them by the experts.

[3] Lyotard seems to be focusing on such a perversion of humanistic science by monied interests in the following passage (Ibid, 46):

> The production of proof, which is in principle only part of an argumentation process designed to win agreement from the addressees of scientific messages, … falls under the control of another language game, in which the goal is no longer truth, but performativity — that is, the best possible input/output equation. The State and/or company must abandon the idealist and humanist narratives of legitimation in order to justify the new goal: in the discourse of today's financial backers of research, the only credible goal is power. Scientists, technicians, and instruments are purchased not to find truth, but to augment power.

the grandest of modernist grand narratives[4] — in the lead role in the model. The more cogent points of the postmodern critique just noted are addressed in the characterization of this project as the development of a *model*. Properly understood, models stand in stark contrast to grand narratives. Unlike grand narratives, models are explicitly tentative, malleable, and limited. They are proposals, suggestions, or "pictures" of how reality might be in certain respects, not dogmatic descriptions of how reality in fact is. Their justification derives from their ability to explain some range of empirical or intuitive phenomena, or to solve philosophical puzzles, not from any sort of extraordinary insight into the nature of reality. But a model's explanations are never complete, and certain questions are inevitably left open. By its nature it is open to modification and extension. It is thus much less prone to become ossified dogma. At the same time, we do not want to losing sight of the fact that models can be *accurate*; that the world can *be* as the model — as far as it goes — *says*. A modeler can claim that one's model is true as far as it goes in proportion to the strength of its justification. An appropriate measure of sensitivity to postmodern concerns needn't, therefore, push one into a skeptical abyss. That noted, we turn to the construction of our model of mathematical ontology.

2 The Dilemma of Abstract Objects

In philosophy, ontology is the study of what there is. Typically, except for extraordinary objects like God, ontology concerns itself not with what there is in particular, but what there is in general. That is, ontology concerns itself with what *kinds* of things exist. For Christians, and theists generally, there are several fairly uncontroversial categories of things: physical things, like planets, trees, strands of DNA, and the like; mental things like pains, thoughts, desires, etc; and spiritual things, understood broadly to include such things as minds, souls,[5] and God. A fourth category is more problematic: abstract things like universals (e.g., **wisdom**, **redness**), relations (e.g., **loving**, **betweenness**), propositions (e.g., **that 7 and 5 are12**, **that God is just**), and, most notably for purposes here, mathematical objects like numbers and sets.

Powerful reasons can be adduced for including abstract objects in one's ontology. Linguists and philosophers of language commonly invoke abstract objects of some ilk or another to serve as the *meanings* of ordinary language statements and the *objects* of

[4]Laplace's haughty dismissal of the "God hypothesis" notwithstanding, the western concept of God is in many ways the fullest expression of the Enlightenment conception of humanity — a free, autonomous, conscious, supremely rational, perfectly moral, all-knowing, individual "self".

[5] Not to suppose that souls are necessarily distinct from minds.

intentional states such as belief, fear, and desire.[6]% % In particular, linguists, philosophers of language, and philosophers of mathematics often take abstract objects to be the meanings of mathematical statements and, more generally, the subject matter of mathematics, the things that mathematics is *about*. And, indeed, it is difficult to see how simple mathematical propositions like **7+5=12** could be true if there is, in no sense, such things as the numbers 5, 7, and 12 and the relation of addition.

Abstract objects raise a dilemma for the Christian. It is fundamental to Christianity that God is the creator of everything other than Godself.[7] Intuitively, however, abstract objects — most of them, anyway — are, like God, eternal; they have always existed and always shall exist. Hence, there is no point at which God could be said to have brought them into being, and hence it is difficult to think of them as creatures.

Moreover, abstract objects — most of them anyway — not only exist eternally, they exist *necessarily*; that is, they could not possibly have *failed* to exist.[8] While there might not in fact have been any red things, still there would have been such a thing as the property of redness. Even if God had decided not to create pebbles and pomegranates and other countable sorts of things, there still would have been the number 11, as well as the proposition that it is prime.

Hence the following dilemma for the theist. Either abstract objects are created by God or they are not. If the former, there are two problems. First, what sense can it even make to talk about things that have always existed and, moreover, that could not have even failed to exist

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[6] See, for example, J. Barwise and J. Perry, *Situations and Attitudes*, Cambridge, MA, MIT Press, 1983; G. Bealer, *Quality and Concept*, Oxford, Oxford University Press, 1982; D. Dowty, R. Wall, and S. Peters, *Introduction to Montague Semantics*, Dordrecht, Holland, Reidel, 1981; J. Katz, *Language and Other Abstract Objects*, E. Zalta, *Intentional Logic and the Metaphysics of Intentionality*.

[7] Probably like the reader, the author cringes at this recent neologism. But, because of various forms of havok and oppression associated with, and often justified by, male imagery for God, he can't bring himself to refer to God with traditional male pronouns any longer. The impersonal gender-neutral pronoun 'it' and its reflexive counterpart 'itself' obviously don't work either. It is hoped that broader adoption of 'Godself' and the like will reduce the heebie-jeebie effect over time. For what it's worth, the author finds that it usually softens the aesthetic blow to read "Godself" in the folksier form "God's own self".

[8] Certain types of abstract object are typically thought of as contingent. Sets that contain contingent objects, for example, are usually themselves taken to be contingent. Any set that contains Willard Quine, for example, would not have existed if Quine hadn't. Similarly, on an influential view suggested initially by Bertrand Russell that features prominently in contemporary discussions in the philosophy language, a proposition that is expressed by a sentence involving a proper name contains the individual denoted by the name as a metaphysical component. Hence, the proposition exists only if the given individual does. Thus, for example, the proposition expressed by 'Quine is a philosopher' would not have existed had Quine not existed. See Russell's 12/12/04 letter to Frege in G. Frege, *Philosophical and Mathematical Correspondence* (Chicago: The University of Chicago Press, 1980), esp. p. 169, and D. Kaplan, "How to Russell a Frege-Church," *Journal of Philosophy* **72** (1975), 716-29, reprinted in M. Loux (ed.), "The Possible and the Actual" (Ithaca: Cornell University Press, 1979) 210-24.

being created? Call this the *coherence* problem. Second, the creation is typically conceived of as *contingent*, the spectacular result of a free and gracious *choice* that God made, something God did not have to do. But if abstract objects are necessary, as noted, they *had* to exist. If so, however, then God had no choice in the matter of their creation. Rather than being the result of a free and gracious act, it seems, God was *constrained* to create. Indeed, because he had no choice, God seems to be constrained by something *other* than God, thus raising further problems about God's autonomy and omnipotence. Call this the *freedom problem*.

But the other horn is no more attractive. Suppose that abstract objects are uncreated. Then, once again, there seem to be two problems. First, if abstract objects are uncreated, then, far from being the maker of all things visible and invisible,[9] it would appear that God is just one more in a vast array of necessary beings existing independently of God's creative power. Call this the *sovereignty problem*. Second, intuitively, a contingent being (i.e., a being that could have failed to exist) needs some sort of explanation for its existence. Since it might have failed to exist, there has to be some reason outside of itself to explain why it exists rather than not. By contrast, a necessary being couldn't but exist. Hence, there is no further need to explain its existence; it exists *a se*: it exists "of and from itself", that is, of its own nature.[10] "Aseity", however, has traditionally been associated with God alone; indeed, some Catholic thinkers have even identified aseity with the divine essence.[11] But if abstract objects also exist *a se*, then God's uniqueness in the universe appears now to be compromised as well. He is just one among many beings that exist of their own nature. Call this the *uniqueness problem*.

Granted, neither horn of the dilemma seems particularly *fatal* to theism. But both force the theist to place severe qualifications on traditional understandings of God's nature, qualifications that most would prefer to avoid. Thus, short of slipping through the horns into nominalism — a difficult alternative that will not be pursued here — the theist must either acquiesce in the dilemma and grasp the least desirable horn, or attempt to show that one horn or the other does not actually compromise God's nature in the manner claimed.

We will grasp the creationist horn in this chapter: it will be argued that abstract objects can be thought to be created in precisely the same sense in which concrete, contingent things are created. Call this view *AO-creationism*.

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[9] Or even most of them, assuming there are only countably many contingent beings.

[10] "Aseity (Latin *a*, from; *se*, itself: *ens a se*) is the Property by which a being exists of and from itself." *The Catholic Encyclopedia*, Volume I, Robert Appleton Company, 1907; Online Edition, 1999 (http://www.newadvent.org/cathen/01774b.htm).

[11] Ibid.

1.1 Continuous Creation

The coherence, sovereignty and uniqueness problems stem in large measure from an inadequate, roughly deistic model of creation. On this model, God brings some initial set of objects into being and sets them a-going. These objects in turn cause other objects to come to be (as in subatomic particle decay, or biological procreation), though, of course, God could presumably intervene at any point in the process and create any additional objects directly *ex nihilo*. But the key idea on this model is that God initially endows whatever objects he does create with a certain measure of "ontological momentum" at the time of their creation that enables them to remain in existence for some period of time, so to speak, under their own power. On this model, it is difficult indeed to make much sense of the idea of an eternal creature of any sort, abstract or otherwise. For if an object has no beginning in time, there cannot *ipso facto* have been a time at which God first endowed it with its measure of ontological momentum; an object that has always been moving of its own momentum can never have been caused to move.

A more adequate model is the so-called "continuous creation" model (CC-model). This model rejects the idea of ontological momentum. Rather, all creatures depend upon God for their existence at every moment, throughout the course of their temporal lifespan; in causal terms, to say that God creates an object *x* is just to say that, at every moment *t* at which *x* exists, God causes *x* to exist at *t* (or *sustains* *x* at *t*, let us say). Creation is thus an on-going act, consisting in God's continuously sustaining all things.[12] Hence, whether or not there happens to be a time at which a thing actually comes to be is irrelevant, as the key element of creation is divine sustenance: non-eternal beings can simply mark a point in time at which God's sustenance of them begins. So the eternal character of abstract objects is no barrier to the coherence of the claim that they are creatures.

But precisely the same reasoning applies to necessary beings. The sense in which God can be said to create a necessary being is no different from the sense in which God is said to be creator in general: for God to create some entity *x* is, once again, simply for God to sustain *x* at every moment of its existence. If *x* is a necessary being, then it is simply the case that God *necessarily* sustains *x* at every moment, that he sustains *x*, so to speak, at all times in every possible world. For a non-necessary being *y*, by contrast, this is simply not the case; rather,

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[12] This doctrine is strongly suggested by St. Paul (Colossians 1:16) and the writer of the Hebrews (1:3), and is at least implicit in St. Thomas (e.g., *Summa Theologica* I, Q. 45, Art. 3). Its most overt expression in the modern period is found in Descartes (*Principles of Philosophy*, 1, XXI). The doctrine has been defended by J. Kvanvig and H. McCann in "Divine Conservation and the Persistence of the World," in Thomas V. Morris (ed.), *Divine and Human Action* (Ithaca: Cornell University Press, 1988, 13-49). Note that it is compatible with this view that other beings *collaborate* with God in acts of creation, in the sense that an object's coming to be might also depend causally, in part, at least, upon the actions of some non-divine agent.

there are possible worlds and (for noneternal beings) times at those worlds at which God does not sustain *y* (and hence at which *y* does not exist).

Hence, on the CC model, the coherence problem evaporates. Conceptually, there is nothing incoherent in the idea of necessary but created beings, and hence, in particular, nothing incoherent about the idea that God creates abstract objects.[13] But if the coherence problem is solved, then the sovereignty and uniqueness problems cannot arise, as they are consequences of the thesis that abstract objects are uncreated. It is worth pointing out that, in particular, on the CC model, the inference from necessity to aseity does not follow. Rather, the idea of objects that, although necessary, do not exist of their own nature but rather whose existence is explained by something external to themselves makes perfect sense. Hence, on this model, nothing other than God exists *a se*, of its own nature. Rather, the existence of everything other than God is — at every moment in every possible world — explained by the exercise of God's creative power.

1.2 Divine Ideas and the Freedom Problem

The freedom problem still remains for the AO-creationist. As a prelude to addressing it, note that the CC model is programmatic only. It establishes the consistency of AO-creationism by providing a sense in which necessary beings can be considered to be created, and that is certainly an important advance. But the thesis at this point seems empty at best. As things stands, it appears that all that the AO-creationist is doing is modifying the traditional, platonic conception of abstract objects: rather than existing *a se*, abstract objects are now asserted to be sustained in existence by God. But nothing about the nature of traditional platonic abstract objects seems to demand this divine sustenance. Thus, a deeper sort of coherence problem remains. What is needed, is a positive model, a reconceptualization, of the nature of abstract objects that, in some sense at least, explains how such an object could be both necessary and created.

The model that will be adopted in this paper is simply an updated and refined version of Augustine's doctrine of divine ideas, a view I will call *theistic activism*, or just *activism*, for

[13] Indeed, that such a state of affairs exists in the Trinity seems at least hinted at by the Nicene Creed's characterization of the Son's being begotten by the Father, and the Spirit's proceeding from the Father and the Son. If either the begetting or the proceeding relation is in any sense causal, then it would follow that, for the Trinitarian, either the Son or the Spirit is, although necessary, ontologically dependent. Of course, this might appear to be skirting heresy, as we have been taking ontological dependence to be a type of *creation*, and this would make the Son or the Spirit a creature. But one could make qualifications that seem to mitigate the heretical worry. For example, one could take begetting and proceeding to be unique types of sustenance in which the divine nature of the sustaining entity or entities is conveyed to the sustained. One could thus acknowledge a sort of necessary dependence within the Godhead without falling into heresy.

short.[14] Very briefly, the idea is this. On this model, abstract objects are taken to be the contents of a certain kind of divine intellective activity in which God is essentially engaged; roughly, they are God's thoughts, concepts, and perhaps certain other products of God's "mental life"). This divine activity is thus causally efficacious: the abstract objects that exist at any given moment, as products of God's mental life, exist *because* God is thinking them;[15] which is just to say that he creates them. Moreover, in the case of non-contingent abstract objects, it is simply the case that God *necesssarily* thinks them, i.e., that God creates them necessarily.

The answer to the freedom problem follows on the activist model stems from the claim that God conceives the abstract objects essentially, i.e., by virtue of God's nature. It is God's power that sustains abstract objects. It is, moreover, not possible that God withhold that power. However, this is only to note that God is essentially rational; God necessarily thinks and conceives, and moreover, necessarily thinks and conceives the same things. Note that this is not to say that God necessarily *believes* (and hence knows) the same things; what God believes will depend in part on contingent facts, e.g., how many people there are at any give moment. However, the *thought* "There are *n* people," for any given *n*, is conceived by God regardless of whether or not it is true. Consider, by analogy, the fact that a grammar of English (ideally) generates all and only the grammatical sentences of English regardless of their truth or falsity. Similarly, God's intellect can be thought of (in part) as a metaphysical "grammar" that "generates" all logically well-formed thoughts and concepts, regardless of their truth or falsity. That God cannot refrain from generating them is no more of an infringement on God's nature than is God's inability to believe a contradiction or commit a sinful act. Being perfectly good, God cannot sin. Likewise, being perfectly rational, God cannot but conceive all logically well-formed thoughts and concepts. They are the natural issue of God's maximally perfect intellect.[16]

On the activist model, then, seen as natural and essential byproducts of God's own intellect, the necessary existence of abstract objects does not seem to imply any kind of

[14] For a detailed exposition and defense of the view, see T. Morris and C. Menzel, "Absolute Creation," *American Philosophical Quarterly* **23** (1986), 353-362; reprinted in Thomas V. Morris, *Anselmian Explorations* (Notre Dame: University of Notre Dame Press, 1987).

[15] Note that we can't analyze the causal relation here counterfactually as simply the claim that if God hadn't thought abstract objects, they wouldn't have existed, since (on the existing semantics for counterfactuals) it is equally true that if abstract objects hadn't existed, God wouldn't have. Despite this *logical* symmetry between God and abstract objects, Morris and I (Ibid.) claim that there is a causal asymmetry.

[16] Admittedly, it does seem to be within God's power, in some abstract sense, to sin, i.e., to bring about some intrinsically evil, morally unwarranted state of affairs. But God will necessarily not exercise that power. Similarly, perhaps it is within God's power, in some abstract sense, to prevent God from conceiving all possible thoughts and concepts. It's just that, necessarily, God won't.

infringement on God's autonomy and omnipotence. Hence the problem of freedom appears to be dissolved.

3 Activism, Numbers, and Sets

The activist model provides a coherent, substantive (if programmatic) account of the sort of activity in which God is engaged that gives rise to abstract objects. On the face of it, though, it is not exactly obvious where mathematical objects fit into the picture. The activist model proposes that abstract objects are the products of God's intellective activity, notably thoughts and concepts. Thoughts, are expressible by sentences, and hence correspond naturally to abstract propositions; and concepts are expressed by nominalizations like 'happiness', 'being a good philosopher', and 'loving' and so correspond naturally to abstract properties and relations. But what about numbers and sets? Traditionally, these entities are not thought of as properties, relations, or propositions (PRPs) of any kind, and hence they find no clear place in the story thus far.

I want to argue that a place can be found, though the search is going to lead us into some fairly deep waters. Let's begin with numbers. A natural view of the numbers, as I will argue below, is that they are properties. For the past century, however, the dominant view of the numbers has been that they are abstract *particulars* of one kind or another. The philosophical roots of this view go back to Frege, in particular, to one of his most distinctive doctrines: that there is an inviolable ontological divide between the denotations (*Bedeutungen*) of predicates, or concepts (*Begriffe*), and the denotations of singular terms, or objects (*Gegenstände*). Taking 'property' to be a loose synonym for 'concept', this doctrine entails that no property can be denoted by a singular term. Since in mathematics the numbers *are* in fact denoted by singular terms, e.g., most saliently, the numerals, it follows that the numbers are not properties but objects, or, loosely once again, particulars.

Now, while few would dispute the inviolability of the distinction between properties and particulars, there are well known and notorious difficulties with the idea that this is simultaneously a distinction between the semantic values of predicates and the semantic values of singular terms.[17] We needn't rehearse these in detail here. For present purposes, let's just note first that our ordinary usage itself doesn't easily square with Frege's doctrine. For there exist a prodigious number of singular terms in natural language that, to all appearances, refer straightforwardly to properties: abstract singular terms ('wisdom', 'redness'), infinitives ('to dance', 'to raise chickens'), gerunds ('being faster than Lance Armstrong', 'running for president'), etc. Now, there are several non-Fregean semantic

[17]Not least of these is Frege's own well known puzzle of the concept **horse**. See "On Concept and Object," in *Translations from the Philosophical Writings of Gottlob Frege*, translated and edited by P. Geach and M. Black (Oxford: Basil Blackwell, 1952), 42-55.

theories in which this fact of natural language is preserved, i.e., theories that are type-free in the sense that difference in ontological type (property, particular, etc.) is not reflected in a difference of semantic type (referent of predicate, referent of singular term, etc.) that prevents singular terms from referring to properties.[18] Hence, we can consistently maintain, *pace* Frege, that at least some properties are the semantic values of both predicates and singular terms. There are thus good reasons for rejecting Frege's semantic doctrine, and hence no cogent reasons for rejecting the idea that numbers are properties *a priori* on Fregean grounds.

There are two plausible accounts that identify the numbers with properties, both of which trace their origins back to the beginnings of contemporary mathematical logic. The first extends back to Frege himself. Frege clearly saw that statements of number typically involve the predication of a numerical property of some kind. For Frege, what is involved is the predication of such a property to a *concept*. Thus, for example, the statement 'There are four moons of Jupiter' is the predication of the property **having four instances** (which is expressed by the quantifier 'There are four') of the concept **moons of Jupiter**. Frege's concept/object doctrine however prevented him from taking this property itself to be the number four, assigning that function rather to its extension.[19] As we've just seen, though, one needn't follow Frege here. In the absence of this doctrine, one is free to make the identification in question, and hence, in general, to take the number *n* to be the property of having *n* instances.[20]

Cantor suggested a different though related view in his (often entirely opaque) discussions of the nature of number. Cantor's insight was that the notion of number is understood most clearly in terms of a special relation between *sets*: we associate the same number with two sets just in case they are "equivalent", i.e., just in case a one-to-one correspondence can be established between the members of the sets. In assigning the same number to two sets then, we are isolating a common property the two sets share; such properties (roughly) Cantor identifies as the cardinal numbers. In his words, the cardinal number of a given set **M** is "the general concept under which fall all and only those sets which

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[18]Cf., e.g., G. Chierchia, "Topics in the Syntax and Semantics of Infinitives and Gerunds," Ph.D. dissertation, Dept. of Linguistics, Univ. of Massachusetts (1984); R. Turner, "A Theory of Properties," *Journal of Symbolic Logic* **55** (1987), 455-472; G. Bealer, *Quality and Concept*.

[19] G. Frege, *Foundations of Arithmetic*, translated by J. L. Austin (Evanston: Northwestern University Press, 1980). See also T. Burge, "Frege on the Extensions of Concepts, from 1884 to 1903," *The Philosophical Review* 93 (1984), 3-34.

[20] This of course can be understood noncircularly in the usual Fregean/Russellian way. This is essentially the analysis of number developed in Bealer, *Quality and Concept*, ch. 6.

are equivalent to the given set."[21] More precisely, the number *n* is a property common to all and only *n*-membered sets, or more simply, the property of having *n* members.[22] Russell presents essentially the same account in *The Principles of Mathematics*, though he ultimately rejects it (unnecessarily, as it happens) because of problems he finds with its nonextensional character.[23]

There are thus at least two ways of understanding the numbers to be properties, both of which are natural and appealing. The first, quasi-Fregean account has a semantical bent, focusing in particular on the predicative nature of statements of number, while the Cantorian account emphasizes the intuitive connection between number and relative size in the more general, abstract form of one-to-one correspondence between sets. Both, however, provide us with good reasons for thinking that numbers are properties of some kind. If so, we have found room for numbers within our theistic framework as it stands.

Now, of course, if we adopt the Cantorian view of number, then it obviously remains to explain just how *sets* fit into our picture. We could avoid this question by choosing instead the quasi-Fregean picture, since it makes no appeal to sets. We will not so choose, however, for two reasons. First, for reasons I will not go into here, I think the Cantorian view is the superior of the two accounts. Second, perhaps more importantly, we want to be able to give our picture the broadest possible scope, and hence we want it to encompass all manner of abstract flora and fauna whose existence platonism might endorse.

So what then are we to say about sets? Are they too assimilable into our framework as it stands? Formally, yes. Both George Bealer and Michael Jubien, for instance, have developed theories in which sets are identified with certain "set-like" properties; roughly, a set {*a*, *b*, …} is taken to be the property **being identical with *a* or being identical with *b* or ….**[24] Given sufficiently powerful property theoretic axioms, one can then show that analogues of the usual axioms of Zermelo-Fraenkel (*ZF*) set theory hold for these set-like properties, and hence that one loses none of *ZF*'s mathematical power.

But this is not an altogether happy move. For instance, as Cocchiarella has noted, intuitively, sets are just not the same sort of thing as properties. Sets are generally thought of as being wholly constituted by their members; a set, "has its being in the objects which belong to it."[25] This conception is deeply at odds with the view of PRPs that underlies metaphysical

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[21] Quoted in M. Hallett, *Cantorian Set Theory and the Limitation of Size* (Oxford: Oxford University Press, 1984), 122. Hallett, I should note, disputes the idea that Cantor held that cardinal numbers are concepts or properties.

[22] For a development and defense of this position, see P. Maddy, "Sets and Numbers," *Nous* **15** (1981), 495-511.

[23] B. Russell, *The Principles of Mathematics* (New York: W. W. Norton, 1937), 112ff.

[24] See Bealer, *Quality and Concept*, ch. 5, and M. Jubien, Models of Property Theory," unpublished ms.

[25] N. Cocchiarella, "Review of Bealer, *Quality and Concept*," *Journal of Symbolic Logic* **51** (1983).

realism, which "are in no sense to be thought of as having their being in the objects which are their instances."[26] Since however it is this very idea of sets as having their being in their members that motivates the axioms of *ZF*, and since this is an inappropriate conception of properties, there seems to be no adequate motivation for a *ZF*-style property theory.

So there is at least ground for suspicion of the thesis that sets are just a species of property. It would be desirable, then, if our account could respect the intuitive distinction between the two sorts of entity. In particular, we should like to trace the origin of sets to a different, and more appropriate, sort of divine activity than that to which we've traced the origin of PRPs. But what sort? Here we have a fairly rich (though often unduly obscure) line of thought to draw upon from the philosophy of set theory. A common idea one often encounters in expositions of the notion of set is that sets are "built up" or "constructed" in some way out of previously given objects.[27] Though generally taken to be no more than a helpful metaphor in explaining the contemporary iterative conception of set, a number of thinkers seem to have endorsed the idea at a somewhat more literal level. Specifically, their writings suggest that sets are the upshots of a certain sort of constructive *mental* activity.

Adumbrations of this idea can be seen in the very origins of set theory. Cantor himself held that the existence of a set was a matter of *thinking* of a plurality as a unity.[28] His distinguished mentor Dedekind is plausibly taken to be embracing a similar line when he writes that

> [i]t very frequently happens that different things … can be considered from a common point of view, can be associated in the mind, and we say they form a *system S*. … Such a system (an aggregate, a manifold, a totality) as an object of our thought is likewise a thing.[29]

Comparable thoughts are expressed by Hausdorff, Fraenkel, and more recently by Schoenfield, Rucker, and Wang.[30] Of these it is Wang who seems to develop the idea most extensively. He writes:

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[26] Ibid.

[27] Cf., e.g., G. Boolos, "The Iterative Conception of Set," *Journal of Philosophy* **68** (1971), 215-231.

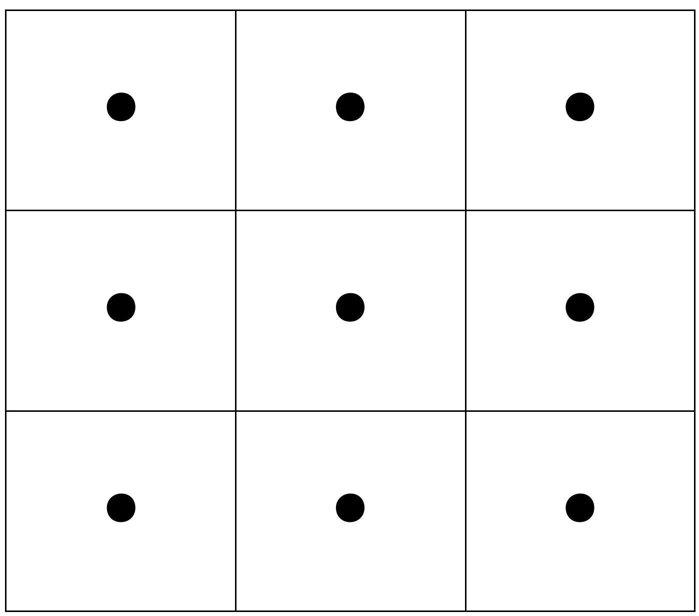
[28] G. Cantor, *Gesammelte Abhandlungen* (Berlin: Springer, 1932), 204.

[29] R. Dedekind *Essays in the Theory of Numbers*, trans. by W. W. Beman (New York: Dover, 1963), 45.

[30] F. Hausdorff, *Grundzuge der Mengenlehre* (Leipzig, von Veit, 1914); A. Fraenkel, *Abstract Set Theory* (Amsterdam: North-Holland, 1961); J. Schoenfield, *Mathematical Logic* (Reading: Addison-Wesley, 1967); R. Rucker, *Infinity and the Mind* (Boston: Birkhauser, 1982); H. Wang, *From Mathematics to Philosophy* (London: Routledge & Kegan Paul, 1974.

> It is a basic feature of reality that there are many things. When a multitude of given objects can be collected together, we arrive at a set. For example, there are two tables in this room. We are ready to view them as given both separately and as a unity, and justify this by pointing to them or looking at them or thinking about them either one after the other or simultaneously. Somehow the viewing of certain objects together suggests a loose link which ties the objects together in our intuition.[31]

I interpret this passage in the following way. Wang here is keying on a basic feature of our cognitive capacities: the ability to selectively direct our attention to certain objects and collect or gather them together mentally, to view them in such a way as to "tie them together in our intuition;" in Cantorian terms, to think of them as a unity. Wang stresses the particular manifestation of this capacity in *perception*, one of a number of related human perceptual capacities emphasized especially by the early Gestalt psychologists.[32] It is best illustrated for our purposes by a simple example. Consider the following array:



Think of the dots as being numbered left to right from 1 to 9, beginning at the upper left hand corner. While focusing on the middle dot 5, it is possible to vary at will which dots in the array stand out in one's visual field (with perhaps the exception of 5 itself), e.g., [1,5,9], [2,4,5,6,8], or even [1,5,8,9]. The dots thus picked out, I take Wang to be saying, are to be understood as the elements of a small "set" existing in the mind of the perceiver.

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[31] *From Mathematics to Philosophy*, 182.

[32] Cf., e.g., W. Köhler, *Gestalt Psychology* (New York: The New American Library, 1947).

The account obviously won't do as it stands. The axiom of extensionality, for instance, seems not to hold on this picture: if you and I direct our attention to the same dots, then each of us has his own "set", despite the fact that they have the same members. Moreover, these entities do not have the requisite ontological stability — they drift in and out of existence with the vagaries of our actual collecting activities. Finally, human cognitive limitations put a severe restriction on the number and size of sets there can be. Wang is well aware of this, and hence builds an account of the nature of sets based on an *idealized* notion of collecting. Irrespective of the success of Wang's efforts, such an idealization, I believe, can be of use to us here in developing a model of divine activity that works sets into our activist framework.

The idea is simple enough: we take sets to be the products of a collecting activity on God's part which we model on our own perceptual collecting capacities. Consider first all the things that are not sets in this sense. While the number of these "first order" objects that we can apprehend at any given time is extremely limited, presumably God suffers from no such limitations; all of them fall under his purview. Furthermore, we can suppose that his awareness is not composed of more or less discrete experiential episodes the way ours is, and hence that he is capable of generating, not just one collection of first order objects at a time, but all *possible* collections of them simultaneously.[33] We suppose next that, once generated, the "second order" products of this collecting activity on first order objects are themselves candidates for "membership" in further collectings, and hence that God can produce also all possible "third order" collections that can be generated out of all the objects of the first two orders. The same of course ought to hold for these latter collections, and for the collections generated from them, and so on, for all finite orders. Finally, in a speculative application of the doctrine of divine infinitude, we postulate that there are no determinate bounds on God's collecting activity, and hence that it extends unbounded through the Cantorian infinite.[34]

Identifying sets with the products of God's collecting activity, then, and supposing that God in fact does all the collecting it is possible for him to do, what we have is a full set theoretic cumulative hierarchy as rich as in any platonic vision. In this way we locate not just the platonist's PRPs, but the entire ontology of sets as well firmly within the mind of God.

4 A Volatile Ontology

Alas, but as is so often wont to be the case in matters such as this, things are more complicated than they appear. The lessons of the last hundred years are clear that caution is

[33] That is, possible in the sense of all the collections he can form from all the nonsets there happen to be; I don't want to suggest that there could be sets containing "merely possible objects," of which I think there are none.

[34] Let me register my awareness that this is no trivial postulate, and that it deserves much further discussion for which there is simply no room here.

to be enjoined in constructing an ontology that includes sets and PRPs. Too easily the abstract scientist, eager to exploit the philosophical power of a platonic ontology, finds himself engaged unwittingly in a metaphysical alchemy in which the rich ore of platonism is transmuted into the worthless dross of inconsistency. Our account thus far is a laboratory ripe for such a transmutation. There are several paradoxes that, on natural assumptions, are generated in the account as it stands.

The first is essentially identical to a paradox originally discovered by Russell and reported in §500 of the *Principles*.[35] In the cumulative or "iterative" picture of sets we've developed here, all the things that are not sets form the basic stuff on top of which the cumulative hierarchy is constructed. Now, the chief intuition behind the iterative picture, one implicit in our theistic model above, is that any available objects can be collected into a set; the "available" objects at any stage form the basis of new sets in the next stage. Thus, since propositions are not sets, all the propositions there are are among the atoms of the cumulative hierarchy; and since all the atoms are available for collecting (God, after all, apprehends them all "prior" to his collecting), there is a set S of all propositions. I take it as a basic logical principle that for any entities x and y and any property **P**,

(∗) if $x \neq y$, then $[\lambda\, \mathbf{P}x] \neq [\lambda\, \mathbf{P}y]$,[36]

i.e., that if x is not identical with y, then the proposition $[\lambda\, \mathbf{P}x]$ that x is **P** is not identical with the proposition $[\lambda\, \mathbf{P}y]$ that y is **P**.[37] Consider then any property you please; the property **SET** of being a set, say. Then by (∗), there is a one-to-one correspondence between Pow(S) (the

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[35] A very similar paradox arises in connection with the notion of a possible world. See M. Loux (ed.), *The Possible and the Actual* (Ithaca: Cornell University Press, 1979), 52-3; P. Grim, "There Is No Set of All Truths," *Analysis* **44** (1984), 206-8; S. Bringsjord, "Are There Set-theoretic Possible Worlds?" *Analysis* 45 (1985), 64; C. Menzel, "On Set Theoretic Possible Worlds," *Analysis* **46** (1986), 68-72; and P. Grim, "On Sets and Worlds: A Reply to Menzel," *Analysis* **46** (1986), 186-191.

[36] This is what can be called a "fine-grainedness" principle that is natural to those conceptions of PRPs that tend to individuate them on psychological grounds, e.g., that properties are identical if it is not possible to conceive of the one without conceiving of the other. Cf., e.g., R. Chisholm, *The First Person* (Minneapolis: University of Minnesota Press, 1981), ch. 1; A. Plantinga, *The Nature of Necessity* (Oxford: Oxford University Press, 1974). For more formal developments cf. Bealer, *Quality and Concept*, and C. Menzel, "A Complete Type-free 'Second-order' Logic and Its Philosophical Foundations," Report No. CSLI-86-40, Center for the Study of Language and Information, Stanford University, 1986. Please note that I will be abusing metalanguage/object language and use/mention distinctions mercilessly throughout this chapter.

[37] Notational remark: $[\lambda x_1 \ldots x_n\ \varphi]$ is an n-place relation that holds among entities $a_1 \ldots a_n$ just in case $\varphi[x_1/a_1 \ldots x_n/a_n]$. Where $n = 0$, this is just the proposition that φ.

power set of S) and the set $T = \{[\lambda\ \mathbf{SET}(s)] : s \in \text{Pow}(S)$. But $T \subseteq S$, since T is a set of propositions, hence $\text{Pow}(S) \preccurlyeq S$,[38] contradicting Cantor's theorem.

A strictly analogous paradox arises for properties (and, in general, relations as well). This is true in particular if one holds that every object a has an *essence*, i.e., the property **being *a***, or perhaps **being identical with *a***.[39] For the same reasons we gave in the case of propositions, properties (and relations) are also among the atoms of the cumulative hierarchy, and hence there is a set M of all properties. Essences being what they are, we have, for any x and y, that

(∗∗) if $x \neq y$, then $\mathbf{E}_x \neq \mathbf{E}_y$,

where $\mathbf{E}_z$ is the essence of z. Consider then $\text{Pow}(M)$. By (∗∗) there is a one-to-one correspondence between $\text{Pow}(M)$ and the set $E = \{\mathbf{E}_z : z \in \text{Pow}(M)\}$. But $E \subseteq M$, since E is a set of properties, hence $\text{Pow}(M) \preccurlyeq M$, contradicting Cantor once again.[40]

There are three quick replies to these related paradoxes to consider. The first is to question the fine-grainedness principles (∗) and (∗∗). Certainly there are views of PRPs on which this would be appropriate. Possible worlds theorists in the tradition of Montague, for example, define PRPs such that they are identical if necessarily coextensional, a "coarse-grained" view incompatible with the fine-grained view we are advocating here. Similarly, views that might be broadly classified as "Aristotelian" hold that properties and relations exist first and foremost "in" the objects that have them, not separate from them, and are "abstracted" somehow by the mind. Such views rarely find any need for PRPs any more fine-grained than are needed to distinguish one state of an object, or one connection between several objects, from another. Whatever the appeal of these alternatives, the problem is that they are out of keeping with our activist model. If we are pushing the idea that PRPs are literally the products of God's conceiving activity, then it would seem that properties which intuitively differ in content, i.e., which are such that grasping one does not entail simultaneously grasping the other, could not be the products of exactly the same intellective activity and hence must be distinct. This is especially pronounced in the cases of singular propositions and essences that "involve" distinct individuals, such as those with which we are concerned in (∗) and (∗∗); it is just not plausible that, e.g., singular propositions "about" distinct individuals could nonetheless be the products of the same activity. To abandon these principles in the context of our present framework, then, would be unpalatable.

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[38] Where $A \preccurlyeq B$ means, in essence, that A is in one-to-one correspondence with a subset of B.

[39] Cf., e.g., Plantinga, *Nature of Necessity*, ch. 5.

[40] Both of these arguments are essentially just special cases of an argument schema that is generally applicable to all n-place relations, $n \geq 0$ (where 0-place relations are propositions).

The second reply is simply to deny the power set axiom. After all, one might argue, many set theorists find the axiom dubious; so why suppose it is true in general, and in the arguments at hand in particular?

The power set axiom has indeed been called into question by mathematical logicians and philosophers of mathematics over the years. The root cause of this disaffection, however, has always been the radically nonconstructive character of the axiom — mathematicians are not in general able to specify any sort of general property or procedure that will enable them to pick out every arbitrary subset of a given set. In this sense, it is the platonic axiom *par excellence*, declaring sets to exist in utter spite of any human capacity to grasp or "construct" them.

It should be clear that any sort of objection on these grounds, as with the previous objection, is just out of place here. For obviously we are far from supposing that set existence has anything whatever to do with human cognitive capacities. Quite the contrary; on our model, the puzzle would rather be how the power set axiom could *not* be true. For supposing that God has collected some set *s*, since each of its members falls under his purview just as the elements of some small finite collection of our own construction fall under ours, how could he not be capable of generating all possible collections that can be formed from members of *s* as well? So this response to the paradoxes is ineffective.[41]

The third reply is that, since there are at least as many propositions (and properties) as there are sets, it is evident that there is no set *S* of all propositions any more than there is a set of all sets; there are just "too many" of them. Hence the argument above breaks down. The same goes for properties, so the second paradox fares no better.

Briefly, the problem with this reply is that how many of a given sort of thing there are in and of itself has nothing whatever to do with whether or not there is a set of those things.[42] The reason there is no set of all sets is not because there are "too many" of them, but rather because there is no "top" to the cumulative hierarchy, no definite point at which no further sets can be constructed. On our model as it stands, however, as nonsets, the propositions and properties there are exist "prior" (in a conceptual sense) to the construction of all the sets. Hence, they are all equally available for membership. But if so, there seems no reason for

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[41] As we'll see below, we can even give the objector the power set axiom, since there are other paradoxes that still arise in the current picture. But I prefer to meet the objector head on here to defend the legitimacy and appropriateness of power set in our activist framework.

[42] I have argued this at length in "On the Iterative Explanation of the Paradoxes," *Philosophical Studies* 49 (1986), 37-61.

denying the existence of the sets *S* and *M* in the paradoxes.[43] So an appeal to how many PRPs there are won't turn back the arguments.

5 A Russell-type Solution

Though always discomfiting, the discovery of paradox needn't necessarily spell disaster. As in the case of set theory, it may rather be an occasion for insight and clarification. Russell's original paradox of naive set theory was grounded in a mistaken conception of the structure of sets that was uncovered with the development of the iterative picture. Perhaps, in the same way, the paradoxes here have taken root in a similar misconception about PRPs. There are two avenues to explore.

The final paragraph in the last section uncovers a crucial assumption at work in the paradoxes: that all PRPs are conceptually prior to the construction of the sets; or again, that all the PRPs there are are among the atoms of the hierarchy. The Russellian will challenge this. He will argue that one cannot so cavalierly divide the world into an ordered hierarchy of sets on the one hand and a logically unstructured domain of nonsets on the other. For although they are not sets, the nonsets too fall into a natural hierarchy of logical *types*. More specifically, in the simple theory of types, concrete and abstract particulars, or "individuals", are the entities of the lowest type, usually designated 'i'. Then, recursively, where $t_1, \ldots, t_n$ are types, let $(t_1, \ldots, t_n)$ be the type of n-place relation that takes entities of these n types as its arguments. So, for example, a property of individuals would be of type (i); a 2-place relation between individuals and properties of individuals would be of type $(i,(i))$; and so on. The type of any entity is thus, in an easily definable sense, higher than the type of any of its possible arguments.[44] By dividing entities whose types are the same height into disjoint levels we arrive at a hierarchy of properties and relations analogous to, but rather more complicated than, the (finite) levels of the cumulative hierarchy.

To wed this conception with our current model we propose that *both* sets and PRPs are built up *together* in the divine intellect so that we have God both constructing new sets *and* conceiving new PRPs in every level of the resulting hierarchy. Thus, at the most basic level are individuals; at the next level God constructs all sets of individuals and conceives all properties and relations that take individuals as arguments; at the next level he constructs all sets of entities of the first two levels and conceives all properties and relations that take entities of the previous (and perhaps both previous) level(s) as arguments; and so on. Thus, since there

[43] Of course, the paradoxes above aside, in such a picture the axiom of replacement would have to be restricted in some way, else replacing on, e.g., the set $\{[\lambda\ \mathbf{SET}(x)] : x \text{ is a set}\}$ would yield the set of all sets. See "On the Iterative Explanation."

[44] Specifically, let the order ord(i) of the basic type i be 0; and if t is the type $(t_1, \ldots, t_n)$, let $\mathrm{ord}(t) = \max(\mathrm{ord}(t_1), \ldots, \mathrm{ord}(t_n))+1$; then we can say that one type t is higher than another t' just in case $\mathrm{ord}(t) > \mathrm{ord}(t')$.

are new PRPs at every level, it is evident that there will be no level at which there occurs, e.g., the set of *all* properties, and hence it seems the paradoxes above can be explained in much the same way as Russell's original paradox.[45]

Easier said than done. Serious impediments stand in the way of implementing these ideas. First of all, there are several well known objections to type theory that are no less cogent here than in other contexts. For example, on a typed conception of PRPs, there can be no universal properties, such as the property of being self-identical, since no properties have all entities in their "range of significance".[46] The closest approximation to them are properties true of everything of a given type. But, thinking in terms of our model, even if many PRPs *are* typed, there seems no reason why God shouldn't also be able to conceive properties whose extensions, and hence whose ranges of significance, include all entities whatsoever.

Along these same lines, type theory also prevents any property from falling within its own range of significance, and in particular it rules out the possibility of self-exemplification. Thus, for example, there can be no such thing as the property of being a property, or of being abstract, but only anemic, typed images of these more robust properties at each level, true only of the properties or abstract entities of the previous level.

Standard problems aside, much more serious problems remain. In many simple type theories, including our brief account above, propositions are omitted altogether. Those that make room for them[47] lump them all together in a single type (quite rightly, in the context of simple type theory). This clearly won't do on the current proposal since the entities of any given type are all at the same level and hence form a set at the next level, thus allowing in sufficient air to revive our first paradox.

A related difficulty is that this proposal is still vulnerable to a modified version of the second paradox as well. Consider any relation that holds between individuals u and sets s of properties of individuals, e.g., the relation $\mathbf{I}$ that holds between u and s just in case u exemplifies some member of s. Let A be the set of all properties of individuals. For each $s \in \mathrm{Pow}(A)$, we have the property $[\lambda x\, \mathbf{I}xs]$ of bearing $\mathbf{I}$ to s. By a simple generalization of the fine-grainedness schemas (∗) and (∗∗), for all $s, s' \in \mathrm{Pow}(A)$ we have that

(∗∗∗) If $s \neq s'$, then $[\lambda x\, \mathbf{I}xs] \neq [\lambda x\, \mathbf{I}xs']$

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45 Note that to pull this off in any sort of formal detail one would have to assign types to sets as well. Since sets on the cumulative picture would be able to contain entities of all finite types, we would also have to move to a transfinite type theory. However, as we will see, the issue is moot.

46 To use Russell's term; see his "Mathematical Logic as Based on the Theory of Types," in J. van Heijenoort (ed.), *From Frege to Gödel* (Cambridge: Harvard University Press, 1967), 161.

47 E.g., the theory in E. Zalta, *Abstract Objects* (Dordrecht: D. Reidel, 1983), ch. 5.

Consider now the set $I = \{[\lambda x\ \mathbf{I}xs] : s \in \text{Pow}(A)\}$. By (∗∗∗) there is a one-to-one correspondence between Pow(A) and I. But $I \subseteq A$, since I is a set of properties of individuals. Hence, Pow(A) $\preccurlyeq A$, contradicting Cantor's theorem.

A little reflection reveals a feature common to both paradoxes that seems to lie at the heart of the difficulty. First, we need an intuitive fix on the idea of (the existence of) one entity *presupposing the availability of* another. The idea we're after is simple: for God to create (i.e., construct or conceive) certain entities, he must have "already" created certain others; the former, that is to say, presuppose the availability of the latter. For sets this is clear. Say that an entity e is a *constituent* of a set s just in case it is a member of the transitive closure of s.[48] Then we can say that a set s presupposes the availability of some entity e just in case e is a constituent of s. For PRPs we need to say a little more. As suggested above, there seems a clear sense in which PRPs, like sets, can be said to have constituents. Thus, a set-like "singleton" property such as [λx x = Kripke] contains Kripke as a constituent. But not just Kripke; for the identity relation too is a part of the property's make-up, or "internal structure;" it is, one might say, a structured composite of those two entities. (We will develop this idea in somewhat more detail shortly.) Combining the two notions of constituency (one for sets, one for PRPs), we can generalize the concept of presupposition to both sets and PRPs: one entity e presupposes the availability of another e^* just in case e^* is a constituent of e.

Now, even though the properties $[\lambda x\ \mathbf{I}xs]$ are properties of individuals, if we look at their internal structure, we see that many of these properties presuppose the availability of entities which themselves presuppose the availability of those very properties, to wit, those properties $\mathbf{P} = [\lambda x\ \mathbf{I}xs]$ such that $\mathbf{P} \in s$. (Analogously for those propositions $\mathbf{p} = [\lambda\ \mathbf{SET}(s)]$ such that $\mathbf{p} \in s$.) Call such properties *self-presupposing*; this notion, independent of the power set axiom, is sufficient for generating Russell-type paradoxes.[49] Conjoined with power set, the possibility of self-presupposing properties can be held responsible for the sort of unrestrained proliferation of PRPs of (in general) any type that fuels the Cantor-style paradoxes as well.

The source of all our paradoxes, then, in broader terms, lies in a failure so far adequately to capture the dependence of complex PRPs on their internal constituents. What we want, then, is a model that is appropriately sensitive to internal structure, but which at the same time does not run afoul of any of the standard problems of type theory.

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[48] I.e., intuitively, just in case it is a member of s, or a member of a member of s, or a member of a member of a member of s, or ….

[49] A similar paradox that doesn't rely on power set is found in Grim, "On Sets and Worlds."

6 A Constructive Solution

Let's review. Our excursion into type theory was prompted by doubts over the idea that PRPs are conceptually prior to the construction of sets. Type theory suggested an alternative: PRPs themselves form a hierarchy analogous to the cumulative hierarchy of sets such that a PRP's place in the hierarchy depends on the kind of arguments it can sensibly take. The idea then was to join the two sorts of hierarchy into one. However, even overlooking the standard problems of type theory, we found that the resulting activist model (-sketch) was still subject to paradox. Our analysis of these paradoxes led us to see that our problems stemmed from the fact that our models were insensitive to the dependence of PRPs on their internal constituents.

How, then, do we capture this dependence? Here we can draw on some recent ideas in logic and metaphysics. Logically complex PRPs are naturally thought of as being "built up" from simpler entities by the application of a variety of logical operations. For example, any two PRPs can be seen as the primary constituents of a further PRP, their conjunction, which is the result of a *conjoining* operation. Thus, in particular, the *conjunction* of two properties **P** and **Q** can be thought of as the relation [λ*xy* **P***x* & **Q***y*] that *a* bears to *b* just in case **P***a* and **Q***b*. A further operation, *reflection*, can be understood to act so as to transform this relation into the property [λ*x* **P***x* & **Q***y*] of having **P** and **Q**. Related operations can be taken to yield complements (e.g., [λ*x* ~**P***x*]), generalizations (e.g., [λ ∃*x***P***x*]), and PRPs that are directly "about" other objects such as our set-like property [λ*x* *x* = Kripke], or the "singular" proposition [λ **PHIL**(Kaplan)] that Kaplan is a philosopher.[50]

On this view, then, the constituents of a complex PRP are simply those entities that are needed to construct the PRP by means of the logical operations, just as the constituents of a set are those entities that are needed to construct the set. It is in this sense that a complex PRP is dependent on its constituents. This then suggests that, analogous to sets on the iterative conception, PRPs are best viewed as internally "well-founded", or at least, noncircular, in the sense that a PRP cannot be one its own constituents.[51]

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[50] These are ideas with syntactic roots in W. V. Quine, "Variables Explained Away," reprinted in his *Selected Logic Papers* (New York: Random House, 1966), and P. Bernays, "Über eine natürliche Erweiterung des Relationenskalkuls," in A. Heyting (ed.), *Constructivity in Mathematics* (Amsterdam: North-Holland, 1959), 1-14, and have close algebraic ties to L. Henkin, D. Monk, and A. Tarski, *Cylindrical Algebras* (Amsterdam: North Holland, 1971). For fuller development within the context of metaphysical realism, cf. Bealer, *Quality and Concept*, Zalta, *Abstract Objects*, and Menzel, "A Completel, Type-free 'Second-order' Logic." Several categories of logical operations have been omitted here to simplify exposition.

[51] This claim is severely called into question in J. Barwise and J. Etchemendy, *The Liar: An Essay on Truth and Circularity*, (Oxford: Oxford University Press, 1987). A full defense of the claim would have to deal at length with their challenge.

This picture of PRPs is especially amenable to activism. For as with set construction, the activist can take the logical operations that yield complex PRPs to be quite literally activities of the divine intellect. This leads us to a further, more adequate model of the creation of abstract entities. At the logically most basic level of creation we find concrete objects and logically simple properties and relations (whatever those may be). The next level consists of (*i*) all the objects of the previous level (this will make the levels cumulative), (*ii*) all sets that can be formed from those objects, and (*iii*) all new PRPs that can be formed by applying the logical operations to those objects. The third level is formed in the same manner from the second. Similarly for all succeeding finite levels. As in our initial models, there seems no reason to think this activity cannot continue into the transfinite. Accordingly, we postulate a "limit" level that contains all the objects created in the finite levels, which itself forms the basis of new, infinite levels. And so it continues on through the Cantorian transfinite.

Now, how do things stand with respect to our paradoxes? As we should hope, they cannot arise on the current model. Consider the first paradox. Since there are new propositions formed at every level of hierarchy, there cannot be a set of all propositions any more than there can be a set of all nonselfmembered sets. Similarly for the second paradox: since essences (as depicted above) contain the objects that exemplify them in their internal structure and hence do not appear to be simple, they too occur arbitrarily high up in the hierarchy and hence also are never collected into a set. What about the two new paradoxes above? The first of these is just a type-theoretic variant on the original paradoxes, and so poses no additional difficulty. And although the concept of self-presupposition can be reconstructed in our type-free framework, the corresponding paradox still cannot arise since there can be no set of all non-self-presupposing properties as the paradox requires.[52] We seem at last to have found our way out of this dense thicket of Cantorian and Russellian paradoxes.

But our task is still not quite complete. Recall that one of our first orders of business was to work (cardinal) numbers into the activist framework. We opted for the Cantorian-inspired view that the numbers are properties shared by equinumerous sets. But just where do they fit into our somewhat more developed picture? Intuitively, numbers seem to be logically simple; they do not appear to be, e.g., conjunctions or generalizations of other PRPs. Hence, they seem to belong down at the bottom of our hierarchy. It follows that there is a set *C* of all numbers at the next level, according to our model. But this supposition, of course, assuming

[52] In type-free terms, for any n+1-place relation **R**, we define the condition $\mathbf{TSP}_{\mathbf{R}}$ such that $\mathbf{TSP}_{\mathbf{R}}(y)$ iff for some z, $y = [\lambda x_1 \ldots x_{i-1} x_{i+1} \ldots x_{n+1}\ \mathbf{R} x_1 \ldots x_{i-1} z x_{i+1} \ldots x_{n+1}]$ and $y \in z$. It is easy to see that, on the present model, nothing — in particular, no n-place relation — satisfies this condition. For any n-place relation of the form $[\lambda x_1 \ldots x_{i-1} x_{i+1} \ldots x_{n+1}\ \mathbf{R}\ x_1 \ldots x_{i-1} s x_{i+1} \ldots x_{n+1}]$, where s is a set, contains s as an internal constituent, and hence on our current model can only have been constructed after the construction of s. Thus, there is no set $s^* = \{y : \sim\mathbf{TSP}_{\mathbf{R}}(y)\}$ (since this would be the universal set) and so no property $y^* = [\lambda x_1 \ldots x_{i-1} x_{i+1} \ldots x_{n+1}\ \mathbf{R} x_1 \ldots x_{i-1} s^* x_{i+1} \ldots x_{n+1}]$.

the truth of the axioms of *ZF*, leads to paradox in a number of ways. For example, one can use the axiom of replacement on C to prove that there is a set of all von Neumann cardinals. I've argued elsewhere[53] that, on certain conceptions of the abstract universe that might allow "overly large" sets, it is appropriate to restrict this axiom to sufficiently "small" sets, and such a restriction would not permit its use here. This would still not redeem the situation, though. On our model, I think we must hold that for every set there exists a definite property which is its cardinal number. For it seems quite impossible that God should construct a set without also conceiving its cardinality, the property it shares with any other set that can be put into one-to-one correspondence with it. Hence, the set C of all numbers must itself have a cardinality k, and so $k \in C$. But it is easy to show (with only unexceptionable uses of replacement) that k is strictly greater than every member of C,[54] and hence that $k > k$\$. Once more we have to confront paradox.

Happily, there is a simple and intuitive solution to this paradox. How many numbers must we say there are? Given our understanding of the numbers as properties of sets, and our reasoning in the previous paragraph, if we divide up the universe of sets according to size, then there must be as many numbers as there are divisions. Numbers are thus in a certain sense dependent on sets in a way that other sorts of properties are not. This suggests a natural way of fitting numbers into our hierarchy in such a way as to avoid paradox: a given number is not introduced into the hierarchy until a set is constructed whose cardinality is that number. The number is then introduced at the next level; God, we might say, doesn't conceive the number until he "has to." Since there are larger and larger sets at every new stage in our hierarchy, there will be no point after which new numbers are no longer introduced, and thus there can be no set of them. Hence, our numerical paradox above cannot get started.

Since our model is informal, the only rigorous way of demonstrating that it is indeed paradox-free is to formalize the picture of the abstract universe it yields and then to prove the consistency of the resulting theory. This can be done. The universe of the activist model can be formalized in a first-order theory that includes all of *ZFC* and a rich logic of PRPs that embodies all the fine-grainedness principles above; and this theory is provably consistent relative to *ZF*.

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[53] In "On the Iterative Explanation of the Paradoxes."

[54] One can, for example, use the theorem (which requires only replacement on ω) that there are arbitrarily large fixed points in the mapping $\aleph$ of (von Neumann) ordinals onto (von Neumann) cardinals. This entails that for any $n \in C$ there is an $m > n$ such that $|\{j \in C : j < m\}| = m$. But $\{j \in C : j < m\} \subseteq C$, hence $|C| = k \geq m > n$.

7 Divine Ideas, Platonic Objects, and Mathematical Knowledge

The possibility of mathematical knowledge has always been a problem for platonism. Knowledge of objects of any ilk seems to require some sort of causal connection between the knower and the objects known. How, then, is mathematical knowledge possible, if it consists in knowledge of nonphysical, causally inert entities like numbers and pure sets? How do we know that our number theoretic and set theoretic axioms describe *those* things? It is not clear, at first sight anyway, that the theistic activist is any better off. For, on the activist view, it would seem that mathematical knowledge consists of knowledge of objects in the mind of God. Such objects seem no more epistemically accessible than classical platonic objects. In response, the activist could postulate that every mind has access to the mind of God. But it seems implausible in the extreme that, say, in first learning of the number 2, a child is accessing the contents of the divine mind. Fortunately, it isn't clear that direct access is needed.

The actualist model builds on the intuition that sets and numbers (being properties of sets) are grounded in our own capacities for perceptual aggregation. Following Wang, we took this capacity to give rise to set-like entities in the mind. There were several reason we could not identify these entities with sets directly, notably, failure of extensionality (as different minds aggregating the same objects give rise to different mental entities), lack of ontological stability, and limitations on the size and number of sets there can be. But these objections are really almost beside the point. For it is not so much the individual constructions themselves that grounds set theory, but their *possibility*. Suppose we are given some urelements. Different agents might all generate set-like constructions by mentally aggregating those urelements. Or they might not. Either way, it doesn't matter. For each such construction is an instance of a more general, overarching idealization — the possibility of such a construction — and *that* is the true object of any set theory based upon those urelements.

Constructivism, on the face of it, seems not to suffer from platonism's epistemological problems. For mathematical objects, if perhaps not *identical* to the products of our own mental activity, are at least intimately tied to those products. Hence, mathematical knowledge can be grounded in the objects of our own construction. But a problem with most all constructivist accounts — both "strict", intuitionistic accounts as well as "looser", classically-based accounts such as those of Kitcher and Chihara[55] — is: what, exactly, is an idealization, the *mere possibility* of a construction? And for whom is the construction possible? For us? For possible humans? For an ideal agent of some ilk? And how do such

[55] P. Kitcher, *The Nature of Mathematical Knowledge* (Oxford: Oxford University Press, 1983); C. Chihara, *Constructibility and Mathematical Existence* (Oxford: Oxford University Press, 1990)

idealizations serve as the subject matter of mathematics? How, in particular, can mathematical statements be *true* if these idealizations do not exist in any sense? How can an existentially quantified statement be true in virtue of the mere possibility of an idealized construction that has not in fact ever been carried out?

The activist model, of course, answers this question by proposing that the idealized constructions that mathematics is about are in fact actual in the divine intellect, and hence that the objects of mathematics can be identified with divine constructions — God's collectings and God's concomitant concepts. In one sense, of course, collectings and concepts in the divine mind are just one more bunch of specific mental constructions. But the identification in question is far from arbitrary. God's is not just one more mind among many, but the fullest possible realization of everything that mind can be and do, and indeed, the very source of mind and consciousness. Moreover, as God is a necessary being, divine constructions have all the stability of existence of traditional platonic objects, and indeed, as God's is the only necessary intellect, they are the only possible constructions with this sort of stability. The identification of the idealized constructions of mathematics with actual divine constructions is thus natural and justifiable. Given both the stability of these objects and their constructive character, this identification satisfies both platonic intuitions about mathematical existence and constructivist intuitions about the connections between mathematical existence and mentality.[56] In addition, however, the activist model seems not to be prone to the same epistemological liabilities as platonism. Baldly, the idea is this. What we do when we construct a set or form a concept is *like* what God does. Hence, our set-like constructions and concepts are like his. We thereby gain basic knowledge of mathematical objects in virtue of knowledge of our own perceptions and concepts, and of their similarity to those in the divine mind.

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[56] That said, it must be noted that the idea that all possible constructions are actual in the divine intellect is, in fact, somewhat problematic. Given the constructive nature of collecting, it always seems possible to collect *more* than what one has already collected. This seems true for the divine intellect no less than for ours. Thus, suppose God has in fact done all the collecting that he can at the current time, so that all the collections there can be (at the current time) are now actual and fixed in the divine intellect. Does it not still seem possible, given the "dynamic" nature of collecting, that God could now *continue* his collecting activity and generate a collection that contains all of the collections that he has in fact generated? If so, then it seems that it makes no sense to say that God, at any given time t, has done all the collecting that it is possible for him to do at t. I will not address this issue in detail here, but perhaps the most promising activist line to pursue here draws upon so-called "reflection" principles. (See, e.g., A. Levy, "Principles of Reflection in Axiomatic Set Theory," *Fundamenta Mathematicae* **49**, 1-10.) Very roughly, these principles can be thought of as saying that, for certain large ordinal numbers κ, the initial segment V_κ of the iterative hierarchy already contains all of the interesting structure of the entire hierarchy. Thought of in terms of our model, such ordinals can be considered natural "stopping points" in the constructing process — in any possible world, God always constructs as far as some natural stopping point.

In a bit more detail, although it is a fertile ground for skepticism, there seem to be good scientific and philosophical grounds for the common sense belief that human beings have similar capacities for organizing and aggregating perceptions and for abstracting concepts. We seem to have observed this in particular in our common ability to aggregate different subgroups of the nine dots in the figure above. Indeed, in a certain clear sense, we can talk about seeing the *same* thing — by which we mean that our numerically distinct perceptions have identical, or at least very similar, structural features. Of course, this cannot be explicitly verified — we cannot literally compare the features of someone else's perceptions with our own. But it is reasonable for us to infer such structural similarity from our respective behavior. We describe our perceptions in similar ways, appear to understand one another when we talk about shifts in figure and ground in an perceptually ambiguous image, and so on. In the same way, we seem to be able to say of one another that we possess the same concept of one thing or another. For example, we all seem to have a concept of a *pair*, or, more abstractly, of *two-ness*. Once again, my concept of two-ness — the thing, or information, or whatever it is in my head that encodes what I understand when I have this concept — is not literally identical with yours. Nonetheless, we can reasonably say that we both have the *same* concept. As before, the grounds for such an assertion will be facts of verbal behavior (we both use the word "two" when we wish to identify a pair) as well as such higher-order facts as that we are both conscious, rational beings with the ability for perceptual discrimination and aggregation as well as conceptual abstraction. And since we can share the "same" perceptions and concepts, we can reasonably be said to have *knowledge* of another's perceptions and concepts.

Admittedly, God presumably does not come to have perceptions and concepts in the same way that we do. Nonetheless, it is a presupposition of most robust brands of theism that we are "created in God's image," i.e., that we are *like* God in certain important respects, particularly with regard to consciousness and rationality. Given this, it seems reasonable to infer, in particular, that our perceptions and concepts are akin to God's. (Granted, we may not want strictly to say that God as *perceptions*, insofar as these depend on some sort of physical perceptual apparatus. But what is important in the notion for purposes here is the idea of *focused awareness* of some ilk, and that certainly seems like a mode of consciousness, however it is realized in fact, that God would possess.) Thus, when we aggregate some objects together in perception or in thought, it is reasonable to say that our resulting collection is the "same" (i.e., has the same members) as one in God's mind. When we abstract a concept of *pair* or *twoness*, it seems reasonable to say that God possesses the "same" concept, a concept with similar content that applies to the same things as our concept, viz., aggregates of two things.

Now, our basic set theoretic axioms are, arguably, based upon facts about, and operations that we can perform mentally on, our set-like perceptions. Insofar as God has the "same" perceptions, then, these same axioms apply to collections in the divine mind. Thus, insofar as the axioms correctly describe our own mental collections they also describe God's. Moreover, intuitively, these axioms describe any possible collections whatever, regardless of size. Consequently, we can be said to have knowledge of the collections in the divine mind for which there are no corresponding collections in any human mind, i.e., according to activism, we have knowledge of *sets per se*. By the same token, our basic number theoretic axioms correctly describe our own concepts. Insofar as God has the "same" concepts, they describe his as well, and hence we can be said to have knowledge of God's number concepts, i.e., according to activism, knowledge of the *numbers per se*. Unlike classical platonism, then, activism seems to provide us with an epistemology that explains our knowledge of mathematics, i.e., of mathematical objects.[57]

8 Conclusion

The postmodernist's skeptical critique of objective knowledge is not so much an argument but a general rejection of a certain philosophical stance, of a certain way of doing philosophy. As we noted in Section 1, there are some important insights behind this rejection. The appropriate response, however, we noted, seemed to be the adoption of a certain tentativeness toward the adoption of any philosophical doctrine. We therefore suggested that the development of philosophical positions, especially in metaphysics, be thought of as the development *models*. For models are by their nature limited, fallible, and revisable.

Like any model, then, the one developed in this chapter is incomplete. There are a number of outstanding issues that have not been addressed. Moreover, the account gives rise to new questions of its own (as described, for example, in footnote 56). However, these caveats noted, insofar as the model does provide an appealing solution to the problem of abstract objects for the believer and again, the postmodernist gives us, at most, reasons to be wary of dogmatic characterizations of its nature — there seems no reason to deny that we cannot reasonably, albeit tentatively, the model as accurate as far as it goes[58].

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[57] The line here is analogous to one developed by Penelope Maddy in "Perception and Mathematical Intuition," *Philosophical Review* **89** (1980), 163-196. Maddy, of course, being a naturalist, does not invoke God in her account. The similarity between the two accounts is in the idea of grounding general set theoretic knowledge in the perception of (or, in my case, the construction in perception of) small, finite collections. Specifically, Maddy argues that sets of concrete objects are themselves concrete, and hence that we have perceptual knowledge of small, finite concrete sets. From this knowledge we extract our basic set theoretic axioms, which then ground our knowledge of sets in general.

[58] Granted, the idea of a model being accurate as far it goes could use some fleshing out that we will not provide here. But the rough idea would be that a model is true as far as it goes if its salient structural features map in a natural way onto the thing being modeled. Thus, a model of an airplane is accurate as far as it goes insofar as whatever represents wings in the model can be mapped in some structure-preserving way to the actual wings

Some believers may not be happy to leave things in such a tentative and uncertain state. But that our beliefs about the world *are* in fact tentative and uncertain is perhaps one of the few themes of postmodern philosophy with which the thoughtful modernist might be forced to agree.

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on the aircraft, and so on for certain other major elements of the actual airplane. Moreover, there ought not to be any additional features of the model that do not correspond to any structural features of the airplane. Lots of things might be missing from the model — that's the "as far as it goes" part. But a model that meets the stated conditions seems reasonably thought of as accurate as far as it goes.